Rotated Grid Formulations

Jian-Guo Li
Met Office, Exeter, EX1 3PB, UK
Email: Jian-Guo.Li@metoffice.gov.uk
December 2013

1. Rotated (Equatorial) latitude-longitude system

Rotated grid is obtained by rotating the North Pole to the new pole position at latitude φ_p and longitude λ_p in the standard latitude-longitude system. The zero-meridian (zero longitude) for the rotated system is chosen to be on the great circle passing both the standard and new poles and on the side opposite to the new pole side, that is, the standard longitude of the new zero-meridian is

$$\lambda_0 = \lambda_p + 180^\circ$$

By setting λ_p within (-180°, 180°), the zero-meridian λ_0 will be within (0, 360°). For instance, if the new 'North Pole' is moved to latitude $\varphi_p = 60^\circ$ and longitude $\lambda_p = -180^\circ$, the new zero-meridian will overlap with the standard zero-meridian, that is, $\lambda_0 = 0$.

Any point A on the globe can be specified by either its standard lat-lon coordinates (λ, φ) or its rotated or equatorial lat-lon (λ', φ'). The relationship between the two coordinate systems may be derived by spherical geometry.

If we choose the new zero-meridian cross-section as the x-z plane with *z* along the standard North Pole and *x* towards the zero-meridian side, the standard and new pole vectors can then be expressed as

$$\mathbf{N} = (0,0,1), \quad \mathbf{N}' = \left(-\cos\varphi_p, 0, \sin\varphi_p\right)$$

The new x and y unit vectors are given by

$$\mathbf{i}' = \left(\sin\varphi_p, 0, \cos\varphi_p\right), \quad \mathbf{j}' = \mathbf{j} = (0,1,0)$$

Assuming the standard longitude difference from the new zero-meridian (or our new x-z plane) to the chosen meridian plane of the point A is

$$\gamma = \lambda - \lambda_0 = \lambda - \lambda_p - 180^\circ$$

where λ is also within (0,360°) as λ_0 and γ within (-180°, 180°), the radial unit vector for point A is then given by

$$\mathbf{A} = \left(\cos\varphi\cos\gamma, \cos\varphi\sin\gamma, \sin\varphi\right)$$

The latitude of the point A in the rotated or equatorial system, φ', is then given by

$$\sin\varphi' = \mathbf{N}'\cdot\mathbf{A} = -\cos\varphi_p\cos\varphi\cos\gamma + \sin\varphi_p\sin\varphi \tag{1}$$

or

$$\varphi' = \arcsin\left(\sin\varphi_p\sin\varphi - \cos\varphi_p\cos\varphi\cos\gamma\right) \tag{1a}$$

The longitude of the point A in the rotated system, λ', can be derived from the invariant relationship of the y component as the rotated system shares the same y axis with the standard system:

$$\cos\varphi'\sin\lambda' = \mathbf{j}'\cdot\mathbf{A} = \cos\varphi\sin\gamma \tag{2}$$

As long as the rotated North Pole is still in the northern hemisphere, the rotated longitude λ' will share the same sign as γ, providing both are within (-180°, 180°). Hence we have

$$\lambda' = \arcsin\left(\sin\gamma \frac{\cos\varphi}{\cos\varphi'} \right) \tag{2a}$$

If cosφ'=0, that is, point A is at the new North Pole, λ' may have any value or simply chosen as zero. This situation is rare as the aim of the rotated grid is to move the new regional domain to be at middle latitude. Note that arcsin yields a range between –π/2 and π/2. Additional information is needed to get the full range longitude in the rotated grid.

Another way to derive the longitude in the rotated system is to use the x' component relation:

$$\cos\varphi' \cos\lambda' = \mathbf{i}' \cdot \mathbf{A} = \cos\varphi_p \sin\varphi + \sin\varphi_p \cos\varphi \cos\gamma \tag{3}$$

If cosφ' is non-zero, that is, point A is not at the new North Pole, λ' is then given by

$$\lambda' = sign(\gamma) \arccos\left(\frac{\cos\varphi_p \sin\varphi + \sin\varphi_p \cos\varphi \cos\gamma}{\cos\varphi'} \right)$$

As arcos returns value from 0 to π, this formula gives the full range longitude in the rotated grid. The rotated longitude shares the same sign as γ. The FORTRAN sign transfer function SIGN(λ', γ) is used in the rotated grid subroutine to make the λ' take the same sign as γ.

2. Transform of standard surface wind to equatorial system

A given wind velocity looks rotated by an angle, α, if viewed from the rotated systems. The angle is equal but opposite to that the local eastern unit vector has turned from the standard system to the rotated system. If we define **i, j, N** as the fixed x, y, z unit vectors in the standard system (x-z plane still coincides with the rotated zero-meridian plane) and **i', j', N**' the unit vectors in the rotated system, the eastern unit vectors at point A can then be represented as:

$$\mathbf{e} = -\mathbf{i}\sin\gamma + \mathbf{j}\cos\gamma, \quad \mathbf{e}' = -\mathbf{i}'\sin\lambda' + \mathbf{j}'\cos\lambda'$$

The rotated unit vectors can be represented in terms of the standard fixed vectors in the vector form:

$$\mathbf{i}' = \mathbf{i}\sin\varphi_p + \mathbf{N}\cos\varphi_p, \quad \mathbf{j}' = \mathbf{j}$$

Substituting them into the eastern unit vector for the rotated system, we have

$$\mathbf{e}' = -\mathbf{i}\sin\lambda'\sin\varphi_p + \mathbf{j}\cos\lambda' - \mathbf{N}\sin\lambda'\cos\varphi_p$$

One of the rotation coefficients is then given by

$$\cos\alpha = \mathbf{e}\cdot\mathbf{e}' = \sin\varphi_p \sin\gamma \sin\lambda' + \cos\gamma\cos\lambda' \tag{4}$$

The northern unit vector at point A in the standard system is given by

$$\mathbf{n} = \mathbf{N}\cos\varphi - \sin\varphi\left(\mathbf{i}\cos\gamma + \mathbf{j}\sin\gamma\right)$$

Because **e**' is rotated clockwise from **e**, the angle between **n** and **e**' is α + π/2. Here the angle α is assumed positive if it is anti-clockwise from the rotated **e**' to the local east **e**. Hence the second rotation coefficient is given by:

$$\sin\alpha = -\mathbf{n}\cdot\mathbf{e}' = -\sin\varphi_p \sin\varphi \sin\lambda' \cos\gamma + \sin\varphi \cos\lambda' \sin\gamma + \cos\varphi_p \cos\varphi \sin\lambda' \quad (5)$$

The second rotation coefficient may also be derived from the first rotation coefficient by

$$\sin\alpha = \pm\sqrt{1-\cos^2\alpha}$$

where the sign of $\sin\alpha$ is taken to be conform to $\sin\lambda'$, that is, positive for $0<\lambda'<180^\circ$. The wind velocity in standard grid should be turned anticlockwise by angle of α as the rotated eastern unit vector **e**' is turned clockwise by the angle α from the standard eastern vector.

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad (6)$$

Output vectors from the rotated system, like the wave spectra, have to be turned clockwise by the same angle in order to restore the true orientation in standard system.

For wave model application in the polar region, the rotated grid of $\varphi_P = 0$ and $\lambda_P = 180^\circ$ is used to provide the fixed map-east direction for wave spectral definition. The map-east is approximated by the rotated local east. The wind rotation angle α is then used to convert the wave spectra between the map-east and local east direction systems. As the rotated latitude and longitude are not used, a single equation to find out the rotation angle will be enough for wave model application. Since $\lambda_P = 180^\circ$ the converted longitude, $\gamma = \lambda$, is equal to unconverted longitude. Inserting $\varphi_P = 0$ and $\lambda_P = 180^\circ$ into the previous equations (1-5), we have

$$\begin{aligned}
&\cos\varphi' \sin\lambda' = \cos\varphi \sin\lambda, \\
&\cos\varphi' \cos\lambda' = \sin\varphi, \\
&\sin\varphi' = -\cos\varphi \cos\lambda, \\
&\cos\alpha = \cos\lambda' \cos\lambda, \\
&\sin\alpha = \cos\lambda' \sin\lambda \sin\varphi + \sin\lambda' \cos\varphi
\end{aligned} \quad (7)$$

Because the rotated grid is used only in the polar region of the standard grid, the rotated latitude φ' will be restricted around the rotated equatorial region, and hence $\cos\varphi' > 0$. The 3rd equation in (7) can be used to evaluate $\cos\varphi'$, and it can be rewritten as

$$\cos\varphi' = \sqrt{1-\cos^2\lambda \cos^2\varphi}$$

Dividing the first two equations in (7) by $\cos\varphi'$ and inserting into the last two, we have

$$\cos\alpha = \frac{\cos\lambda \sin\varphi}{\cos\varphi'}, \qquad \sin\alpha = \frac{\sin\lambda}{\cos\varphi'} \quad (8)$$

Or

$$\cos\alpha = \frac{\cos\lambda \sin\varphi}{\sqrt{1-(\cos\lambda\cos\varphi)^2}}, \qquad \sin\alpha = \frac{\sin\lambda}{\sqrt{1-(\cos\lambda\cos\varphi)^2}}, \quad (8a)$$

Although the sine function is simpler, it is not practical to retrieve the full range rotation angle because arcsine usually returns value from $-\pi/2$ to $\pi/2$. The arccosine function, however, returns value from 0 to π but makes the sign ambiguous. So it is better to use $A=\arccos(\cos\alpha)$ to derive the angle absolute value and the first equation in (7) to determine the sign because α shares the same sign as $\sin\lambda'$.

$$\alpha = \mathrm{sgn}(\cos\varphi \sin\lambda)\arccos\left[\frac{\cos\lambda \sin\varphi}{\sqrt{1-(\cos\lambda\cos\varphi)^2}}\right] \tag{9}$$

The rotation angle becomes ambiguous at the North Pole because cos φ = 0 and λ is not defined. Because the map-east coincides with the local east on the zero-meridian, the rotation angle for the North Pole should be chosen as zero, or α = 0. If the longitude at the North Pole is set as λ = 0, and the function sgn returns a positive sign for a zero value (this is the case in FORTRAN but may not be true in other languages, such as IDL), Eq. (9) can produce the desired value α = 0. It is safe to set α = 0 for the North Pole if you are not sure of the subtle difference of the sign function in different languages. Eq. (9) can be verified with the rotation angles of any points close to the Pole along four meridians: λ=0º, 90º, 180º and -90º. The rotation angle should be approximately α=0º, 90º, ±180º, and -90º, respectively.

For Arctic wave propagation, the map-east is used within the Arctic region for definition of the wave spectrum. Hence, the Arctic spectrum has to be rotated clockwise by the angle α before it is used as a local-east reference spectrum (in the global part northern boundary). That is, the spectral component at angle θ' reference to the map-east becomes the spectral component of angle θ = θ' – α in relative to the local east. Similarly the global spectrum at angle θ from the local east becomes the spectral component of angle θ' = θ + α in the map-east system (for Arctic part boundary condition).

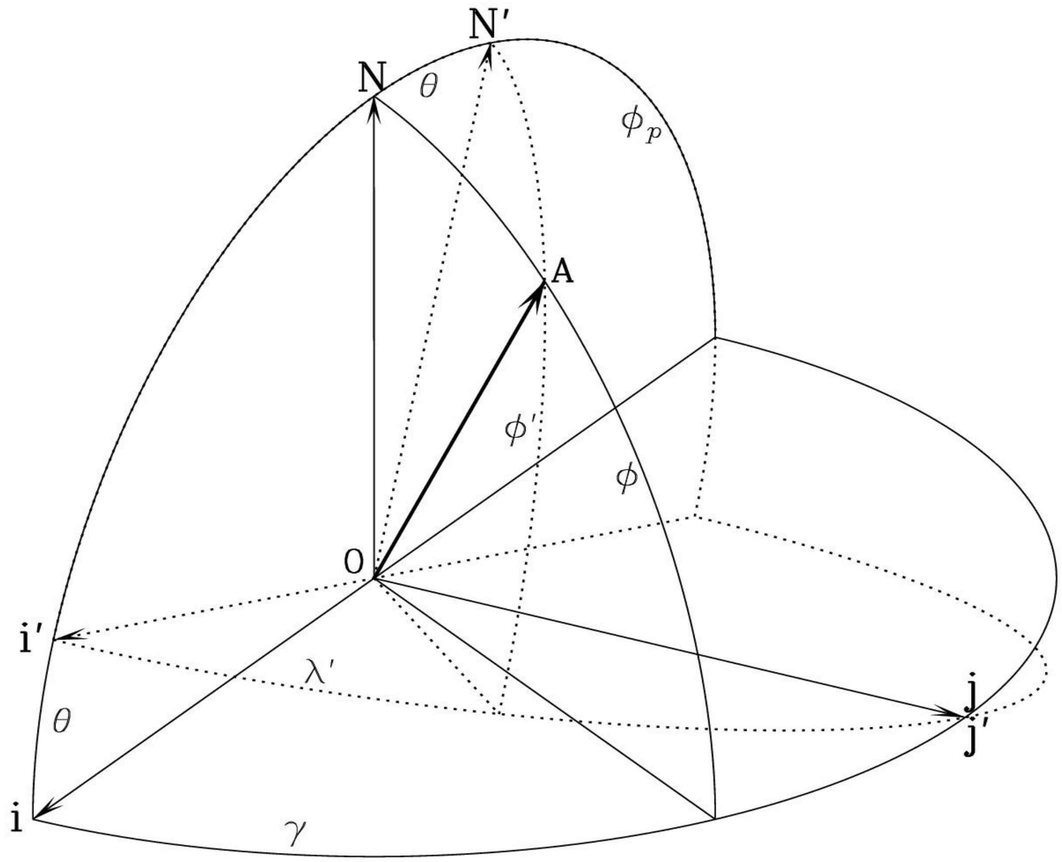

Fig.1. Illustration of the relationship between the standard and rotated latitude-longitude systems.

3. Reversed transformation from equatorial to standard system

For reverse translation from equatorial latitude-longitude to standard latitude-longitude, the above formulations may be used as if the rotated pole is at the same latitude φ_p'= φ_p and longitude $\lambda'_{p=}$ 0°.

The transform will then convert the rotated lat-lon to standard lat-lon except that the longitude is measured from the new zero-meridian of the standard system at λ'_0=180°, which is the original rotated pole longitude λ_p in the standard system. Simply add the original rotated pole longitude will complete the transformation. So the reverse formulations are obtained by the following substitutions in (1-4):

$$\gamma \to \lambda' - 180^\circ, \qquad \lambda' \to \lambda - \lambda_p$$
$$\varphi \to \varphi', \qquad \varphi' \to \varphi$$

Hence we have

$$\sin\varphi = \sin\varphi_p \sin\varphi' + \cos\varphi_p \cos\varphi' \cos\lambda' \tag{6}$$

$$\cos\varphi \sin\left(\lambda - \lambda_p\right) = -\sin\lambda' \cos\varphi' \tag{7}$$

The sign of λ-λ_p will be opposite to λ' as the new zero-meridian is 180° from the old zero-meridian. Equivalently, multiplying by -1 and noting $\lambda_0 = \lambda_p + 180°$, it becomes

$$\lambda - \lambda_0 = \arcsin\left(\sin\lambda' \frac{\cos\varphi'}{\cos\varphi}\right)$$

The sign of λ-λ_0 will be the same as λ'. If $\cos\varphi$=0, that is, at the standard pole, λ may take any value or simply assume to be zero.

The reverse rotation angle for the wind vectors will be just the opposite of the original angle. So simply reverse the sign of α in (4) for conversion of wind velocity from rotated system to the standard system, that is,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \tag{8}$$

4. Rotation of wave spectra between standard and equatorial systems

Wave spectra will be rotated the same angle as the wind velocity. As wave spectra are represented as discrete directional arrays in the wave model, the rotation will be divided into a simple re-numbering of directional bins and shifting of wave energy by the fractional angle (remainder of the rotation angle divided by the bin width, hence less than one directional bin width). Wave energy will be assumed to be uniform within one directional bin, so the fractional shift will be simply passing the fractional energy from the 'upstream' bin to the lower one. The equal amount will be deducted from the upstream bin so the total wave energy is conserved and remain positive-definite (in analogous to the first order upstream advection scheme).

The North Atlantic and European (NAE) model is used as an example to demonstrate the rotated grid and the wave spectral rotation. The NAE model domain is set to be in a rotated grid with the new pole at PLAT=37.5, PLON=177.5. The mesh is uniform lat-lon 0.11 deg in the rotated grid with 720 longitude points from 313.02 and 432 latitude points from -22.49. The rotated grid point locations in standard grid are showed Fig.2:

Wave spectra at lateral boundaries from the standard grid model have to be rotated before being used as boundary conditions in the rotated grid. A wave spectral rotation program is provided in the WAVEWATCH III model. One spectral rotation example is illustrated in Fig.3. The wave spectral is a typical cosine square distributed JONSWAP spectrum with 25 spectral bins and 24 directions. The peak frequency is about 0.1 Hz and the mean direction is about 45 degree from the north in the northeast (central panel in Fig.3). The spectrum is rotated anti-clockwise by 80 degree and is shown in the left panel in Fig.3. The right panel in Fig.3 compares the 1-D spectra of the original and rotated 2-D spectra. They are identical as expected because rotation does not affect the 1-D spectra.

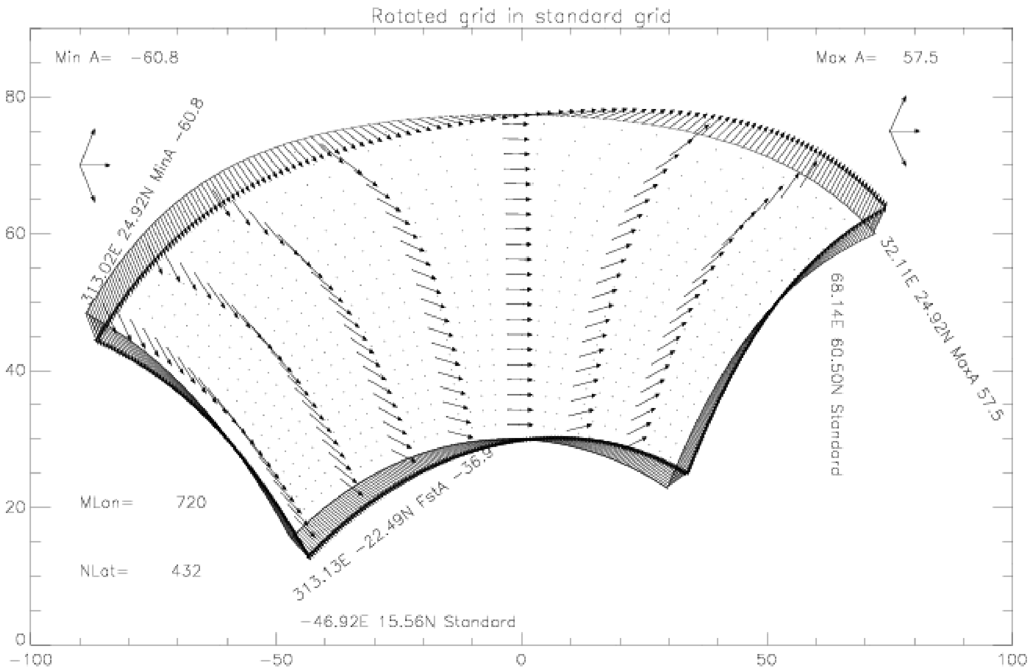

Fig.2. The rotated NAE domain plotted in the standard lat-lon system. The wind velocity arrows are rotated pure westerly wind in the standard system and plotted in reference to the standard system, that is, what a true westerly wind looks like in the rotated system.

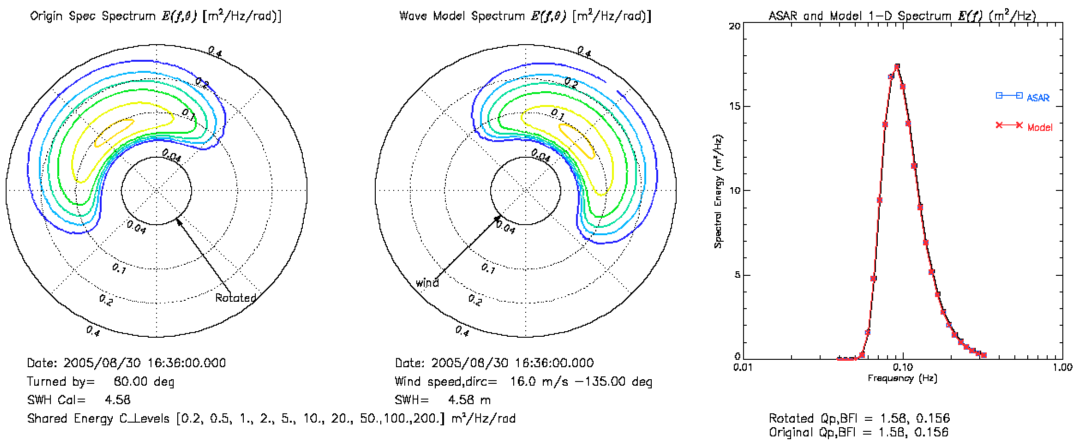

Fig.3. Rotation of wave spectra for rotated system. The original spectrum is shown in the middle panel and the rotated spectrum by the left. The right panel compares the 1-D spectra of the original and rotated 2-D spectra.